

Localized quasi-(bi)harmonic field and its applications

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Abstract

(Bi)harmonic field has wide applications in geometry processing. Traditionally, to locally control the influence region of a (bi)harmonic field, users usually need to determine the range of its support, regions with non-zero scalar values, by prescribing appropriate boundary conditions. However, this way is non-intuitive and inconvenient. We proposed localized quasi-(bi)harmonic field, which is achieved through a ℓ_1 -norm regularized convex optimization. It can conveniently control the local support of the scalar field while still keeping some nice properties of the (bi)harmonic field. We applied the localized quasi-(bi)harmonic field in applications such as shape deformation and shape merging, and the experiment results show its benefits.

Key words: Harmonic field, Quasi-(bi)harmonic field, Local control, Shape deformation, Shape merging

1. Introduction

Many geometry processing problems can be reduced to the problem of computing certain scalar field defined on surfaces or volumes. Harmonic field, as the solution of the Laplace equation with given boundary conditions, is one of the most popular scalar fields. It has widely applications nearly covering the whole geometry processing field, ranging from interpolation, modeling, filtering, parameterization, remeshing, deformation, segmentation, morphing, to many more. This may benefit from its nice properties, such as smooth, unique up to the boundary conditions, with no local extrema other than on the boundary, a measurement of mapping distortion, efficient to calculate, etc.

Harmonic field is a global support scalar field. Given Dirichlet's boundary condition, a harmonic field smoothly blends the boundary values across the whole domain. To design such a field, users need to specify appropriate boundary conditions according to their requirements, which is the key step for the geometry processing applications. In quad-meshing or hex-meshing oriented parameterizations, one type of method is to generate quad-patch boundaries or polycube surfaces as the boundary conditions (Tong et al, 2006; Dong et al, 2006;Li et al, 2007; Xia et al, 2010; Li et al, 2011). In mesh segmentation, one usually places boundary conditions on the prominent features of the shape (Lai et al, 2009; Tierny et al, 2012; Au et al, 2012). However, in some applications, it is not intuitive for users to determine such boundary conditions. For example, in handle based deformations (Botsch and Kobbelt, 2004; Zayer et al, 2005), one should select and set handle vertices as the boundary conditions to control the influence region of the deformation, which is a difficult and inconvenient task. In Poisson-based merging (Yu et al, 2004; Deng et al, 2013), it is not easy to determine the influence region of the error difference as well, or it may excessively change the shape if the influence region is not prescribed.

In the paper, we propose a new scalar field, which can control the influence region conveniently while inherits some nice properties of harmonic field. The scalar field is smooth with controllable zero-value region away from the boundary and is called "localized quasi-(bi)harmonic field". To calculate the field, we formulate the problem as a ℓ_1 -based convex optimization (Xu et al, 2015) and solve it via a quadratic program. We further apply the scalar field to mesh deformation and mesh merging and show its benefits of controllable influence region over the traditional algorithms.

The rest of the paper is organized as follows: Section 2 is the related work. Section 3 reviews the harmonic field.

Section 4 introduces our localized quasi-(bi)harmonic field and its solution. Section 5 shows results of localized quasi-(bi)harmonic field applied to mesh deformation and merging. And the last section makes a conclusion.

2. Related work

Harmonic map or harmonic field is widespread in the geometry processing community. It appeared in early days to compute minimal surfaces and was discretized on triangular mesh (Pinkall et al, 1993) and henceforth other domains, such as tetrahedral mesh (Wang et al, 2004) and general polygonal meshes (Alexa and Wardetzky, 2011).

The energy of the harmonic field (Dirichlet energy) is a measurement of the stretching of a mapping, which can be used to compute parameterizations and deformations. Eck et al (1995) first proposed harmonic mapping for surface mesh parameterizations. Gu et al (2003) computed global conformal parameterizations for surfaces with non-trivial topologies, which reduces to calculating harmonic fields. Wang et al (2004) utilized harmonic field to 3D for harmonic volumetric parameterizations, which can be extended to other kinds of volumetric mappings (Jin et al, 2015). Joshi et al (2007) proposed harmonic coordinates achieved by harmonic fields for cage deformations. Ben-Chen et al (2009) introduced a variational framework to compute harmonic maps for shape deformations. Deng et al (2013) recently used MVC (Mean Value Coordinates) method to approximate harmonic fields for mesh merging.

Harmonic field is a smooth scalar field which can be obtained efficiently, thus it can be used for approximating distance field, remeshing, etc.. Ni et al (2004) took harmonic field as a fair Morse function to extract the topological structure of a shape. Zayer et al (2005) used harmonic field instead of geodesic distance field to propagate rotations for shape deformation and the method was extended to the volumetric case by Liao et al (2009). Dong et al (2005) applied harmonic field for quadrilateral remeshing by computing its gradient field and its orthogonal vector fields. Tong et al (2006) used singularity graph to partition the mesh into quad-patches and computed global harmonic fields with constraints on each patch boundary for quad-remeshing. Xia et al (2010) decomposed a volume into the direct product of a two-dimensional surface and a one-dimensional curve and traced the integral curve along the harmonic field for hexmeshing. Li et al (2011) also computed three orthogonal harmonic fields for facial hex-meshing.

Harmonic field is shape-aware and could be used for shape analysis. Lai et al (2009) extended the random walk method in image segmentation to mesh, which reduces to computing some harmonic fields. Tierny et al (2012) designed a harmonic field as a Morse function and extracted its Reeb graph to segment the mesh into patches for quad-remeshing. Au et al (2012) designed a concavity-aware harmonic field which is befit for segmentation. Zheng et al (2013) used harmonic field for pairwise analysis and induced a new class of shape descriptors that are more global, discriminative, and can effectively capture the variations in the underlying geometry.

As is introduced, harmonic field is quite useful. To design a harmonic field, one key step is to prescribe the boundary conditions. However, it is not an easy task for users in some situations, e.g. controlling the influence region, which is vital in some applications such as mesh deformation and mesh merging. To the best of our knowledge, we have not seen any work on designing harmonic field with local support in the field of geometry processing. But some work on Poisson-based image cloning devised special harmonic field with control of influence region to preserve the important content of the source image. Ding and Tong (2010) used an alpha map to indicate the region to be preserved. Du and Jin (2013) substituted a weighted gradient of the source and destination image with the traditional gradient and the weight is used to control the influence region. In geometry processing, to achieve the goal of locally controlling the deformation, Deng et al (2013) proposed a sparsity-enhancing framework for exploring local modifications of constrained meshes, which has similarity with our work. However they focused on computing deformations with local control while ours aiming at computing a scalar field. Zhang et al (2014) presented local barycentric coordinates via optimizing the total variation of the barycentric coordinate functions which is also used to locally deform a shape. Our work reformulates the harmonic field from the sparse-coding point of view, enabling it to flexibly control the influence region with one parameter while not sacrificing its good property, e.g. smoothness.

3. Harmonic field

Harmonic field is the solution of the following steady-state elliptic equation with certain boundary conditions,

$$\Delta f = 0, \qquad (1)$$

where Δ is the Laplace operator and *f* is a twice continuously differentiable real function defined on domain Ω . The above equation is also known as Laplacian equation, which can be derived through the critical point of the Dirichlet energy below,

$$E_d = \int_{\Omega} |\nabla f|^2.$$
 (2)

For the discrete setting, we consider the case when the harmonic function is defined on a triangular mesh $M = \{V, T\}$, where $V = \{v_i\}$ denotes the vertex set and $T = \{t_i\}$ stands for the set of triangle faces. Due to the piecewise-linear nature of the triangular mesh, we can uniquely determine a scalar field *u* by prescribing a scalar value u_i for each vertex v_i and extend linearly over each triangle. The density of Dirichlet energy on each triangle face is thus constant and the total energy over the triangular mesh can be calculated as,

$$E_{d} = \sum_{t=1}^{|T|} A_{t} \| \nabla_{t} u \|^{2}, \quad (3)$$

where A_t is the area of triangle *t*. By the derivation the above energy function and setting its gradient to zero, we can obtain a system of linear equations (Pinkall et al, 1993),

$$Lu = 0,$$
 (4)
where $u = [u_1, u_2, ..., u_N]^t$ $(N = |V|)$ and $L = (l_{ij})$ is the sparse Laplacian matrix whose entries are given by,

$$\begin{cases} \sum_{\langle i,k\rangle\in E} \omega_{ik} & \text{if } i=j \\ -\omega_{ij} & \text{if}\langle i,j\rangle\in E \\ 0 & \text{else} \end{cases}$$
(5)

In the above equation, *E* is the edge set of the mesh, ω_{ij} can be set according to different discretization schemes and applications (Wardetzky et al 2007), and it is usually defined by the well-known geometry-aware cotangent weights,

$$\omega_{ij} = \frac{1}{2} (\cot(\alpha_{ij}) + \cot(\beta_{ij})),$$

where α_{ij} and β_{ij} are opposite angles to edge $\langle i, j \rangle$.

Combining Eq. (4) with the Dirichlet's boundary constraints represented as the matrix equation, Cu = p (where p is the vector of boundary values), the harmonic field can be calculated through a symmetric and positive-definite linear system, which can be efficiently solved by Cholesky factorization.

4. Localized Quasi-(bi)harmonic Field

Harmonic field is uniquely determined by the Dirichlet's boundary conditions. In some applications, users need to carefully set boundary conditions appropriately to control the influence region of the field, such that the field becomes local support. However, this method is not intuitive and inconvenient.

We therefore proposed a new scalar field which can flexibly control the influence region. The key idea lies on the observation that local support means the values of the field in the remaining domain are zeros, and thus controlling the influence region equivalents to controlling the number of zero-value of the field u, which can be formulated by the sparse coding theory (Xu et al, 2015).

In sparse coding theory, $||u||_0$ is the direct choice of counting the number of non-zero entries of the vector u, where $|| \cdot ||_0$ denotes the ℓ_0 -norm. However, the problem of finding the sparsest solution using $|| \cdot ||_0$ is NP-hard and difficult even to approximate. In common approach, the ℓ_0 -minimization problem is usually relaxed to ℓ_1 -minimization, which also induces sparsity but becomes convex. Thus, to achieve a sparse harmonic field, we obtain the following equation,

 $\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} ||\boldsymbol{u}||_1$

s.t.
$$Lu = 0, Cu = p,$$
 (6)

Note that the feasible region described by the equalities in Eq. (6) is a single point, thus the sparsity measure of the object function does not take effect. We therefore relax the Laplace equation Lu = 0 to $||Lu||_2 < \varepsilon$, where ε is a non-

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negative real parameter, and transform Eq. (6) to the following equation,

 $\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} ||\boldsymbol{L}\boldsymbol{u}||^2 + \lambda ||\boldsymbol{u}||_1,$

s.t.
$$Cu = p$$
 (7)

where λ is a positive parameter balancing the two terms.

The formulation above is the well-known least absolute shrinkage and selection operator (LASSO) (Tibshirani, 1996). To solve the problem, we eliminate the absolute sign in Eq. (7) by introducing a set of slack variables, $z = [z_1 \ z_2 \ \dots \ z_N]$, where *N* is the number of vertices. Then it can be transformed to the quadratic program as follows,

 $u^* = \arg \min_{u,z} u^T L^T u + \lambda z$,

s.t. Cu = p, $-z_i < u_i < z_i \ i = 1, 2, ..., N$ (8)

Note that the first term of the object function in the above equation is the thin-plate energy (Kobbelt et al, 1998) and its critical point is a biharmonic field. Because the Laplace equation Lu = 0 is not strictly satisfied in Eq. (8), we call its solution "localized quasi-biharmonic field". In practice, we could replace the thin-plate energy with the Dirichlet energy $u^{T}Lu$ (also known as "membrane energy") and its solution thus corresponds to "localized quasi-harmonic field". The optimization is convex, i.e. global minimizer always exists, and standard solvers are available.

5. Results

5.1 Fields Comparison

We solved the constrained optimization in Eq. (8) by the MOSEK optimizer (ApS, 2015). Fig. 1 shows the results of the (bi)harmonic fields and localized quasi-(bi)harmonic fields. Note that all the fields smoothly blend from one end to the other. However, the (bi)harmonic field needs to constrain vertices on both ends of the rectangle and the influence regions cover the whole domain, while the localized quasi-(bi)harmonic field only constrains vertices on one end and their influence regions are local. Besides, comparing to the quasi-biharmonic field, the quasi-harmonic field has much narrower influence regions. Thus we could choose one of them for different applications.



Figure 1: (Bi)harmonic field and localized quasi-(bi)harmonic field. (a) harmonic field, (b) biharmonic field, (c) localized quasi-harmonic field ($\lambda = 1$), (d) localized quasi-biharmonic field ($\lambda = 1$). The green points denote the constrained vertices, where the right ends of all the bars are set to 1 and the left ends of the bars (a) and (b) are set to 0. The colors from blue to red map the scalar value from small to large.

Our proposed field shares the nice property of smoothness with the traditional (bi)harmonic fields. Though it is not easy work to give a theoretical proof that a single local support exists only near the constrained region, we found the fact by experiments. One possible explanation is that the membrane or the thin-plate energy tries to smooth the field. And the appearance of other non-simply connected support would lead to non-smooth field, which may be restrained by the energy minimization.

5.2 Applications

To show the benefits of the proposed localized quasi-(bi)harmonic field on controlling the influence region, we experimented on two applications, handle-based mesh deformation and Poisson-based mesh merging.

In handle-based mesh deformation, the position change of handle vertices are propagated to the whole mesh by a smooth field, i.e. (bi)harmonic field. The filed can be achieved by solving a Laplace equation or its higher order version, $L^{(n)} u = 0$ (*u* is the $3 \times N$ matrix for the coordinates of all the vertices), under Dirichlet's boundary conditions. Thus the new shape can be recovered as v' = v + u (Botsch and Kobbelt 2004), where *v* is the rest shape, and *v*' is the new shape.

Commonly, to generate a deformation field, users need to prescribe several handle constraints as boundary conditions. However it is non-intuitive, because users usually do not know which vertices should be fixed. While for our localized quasi-(bi)harmonic field, users could save the effort to select the fixed vertices.



Figure 2: Planar deformation with localized quasi-biharmonic field (b) and biharmonic field with different constraints (c, d, e). (a) is the rest shape. The color indicates the displacement from the rest pose (the same below).

Our method can deform shapes locally by manipulating the handles only. Fig. (2) and Fig. (3) show the comparison results of these two fields for the cases of planar deformation and surface deformation. In Fig. (2), with only one handle vertex, the result of localized quasi-biharmonic field achieves its goal in locally changing the shape (b), while the result of biharmonic field just parallel translates the shape without being deformed. To obtain the local deformation by biharmonic field, we set different positional constraints and obtained the results in (d) and (e). However, in both cases, the shape changes globally. Fig. (3) displays another example, where the feet and the tail of the dinosaur model are nearly fixed after the deformation with our method, though only the mouth is constrained, while we have to specify the fixed region to achieve the similar results for the traditional method.



Figure 3: Surface deformation for the dinosaur model with biharmonic field where the mouth and the feet are constrained (a) and localized quasi-biharmonic field where only the mouth is constrained (b).

Poisson-based mesh merging is first proposed by Yu et al (2004). The goal is to merge two partial meshes at their open boundaries while keeping the detail of the two meshes. The key idea of Poisson-based method is to seek a function f, whose gradient field is as close as possible to that of the source mesh S, which is formulated as $\min_{f} || \nabla f \cdot Rg ||$, where g is the gradient field of S and R is a rotation field. The minimization leads to a Poisson equation, $\Delta f = div(Rg)$. If the boundaries of the two meshes are well-aligned, R can be approximately given by the identity matrix I. Thus the Poisson equation becomes a simpler form, $\Delta f = div(g)$.

Following the idea of Pérez et al (2003), we define the correction function f and transform the above Poisson equation to the Laplace equation,

$$\Delta f = 0$$
 s.t. $f|_{\partial\Omega} = (f^* - h)|_{\partial S}$

where f^* and h are the coordinates of the target and source mesh. And the final output is f = h + f. The new formulation can be also explained as to propagate the differences on the boundary to the inner mesh.

If the boundary differences are large, the whole shape to be merged may be distorted, leading to visual artifacts for the

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traditional harmonic field, while the most part of the shape keeps unchanged and the result looks more natural for our field. Fig. (4) and Fig. (5) show two merging examples with the harmonic field and the localized quasi-harmonic field. Fig. (4) merges a dragon head to a horse body. Though the results are visually the same, our result has less displacement error than that of the traditional harmonic field (see the mouth of the dragon head). Fig. (5) merges a face mesh to a head mesh. Note the shape of mouth is largely twisted with the traditional method while ours is well preserved (see the figures in (b) and (c)).



Figure 4: A dragon head is merged to a horse body with the harmonic field (b) and the localized quasi-harmonic field (c). The initial positions and merging boundaries of the both meshes can be seen in (a).



Figure 5: A face is merged to a head with the harmonic field (b) and the localized quasi-harmonic field (c). The initial positions and merging boundaries of the both meshes are shown in (a).

5.3 Parameter

The only parameter λ in Eq. (7) is used to control the size of influence region and the smoothness of the field. The value of λ is inversely proportional to the size of influence region, i.e. larger values lead to smaller influence region and vice versa. Fig. (6) displays deformation results with different values of λ . In the figure, we find that with the decreasing of λ , the size of influence region of the deformation increases, and the deformed shape becomes smoother. Note that the handle of the ball in (a) is very sharp because of the large value of λ . Thus, we have to choose an appropriate value for λ to balance the size of influence regions and smoothness for different applications.



Figure 6: A sphere is deformed by localized quasi-biharmonic field with different parameters λ . (a) $\lambda = 10$, (b) $\lambda = 1$, (c) $\lambda = 0.1$,(d) $\lambda = 0.01$.

5.4 Performance

We implemented the algorithm in C++ and ran it on a computer with 2.33-GHz 4 cores CPU and 4-GB memory. All

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linear algebra operations were carried out with Eigen library (Guennebaud et al, 2010). In the optimization (Eq.7), the quadratic program has $6 \times N$ variables and $6 \times N$ linear inequalities for the both applications and it was solved by MOSEK optimizer. Tab. (1) presents time statistics of the results above . We note that the performance is not so good for interactive applications in general.

-	Mesh	(#V/#F)	Times (s)	Mesh	(#V/#F)	Times (s)
_	Woody	(694/1267)	5.9	Dragon	(2734/5429)	107.8
_	Dina	(4434/8864)	740.8	Face	(4708/9162)	536.6

Table 1: Time statistics for the results, where #V/#F stand for the number of vertices and faces of the mesh.

6. Conclusion

We proposed a new scalar field, called localized quasi-(bi)harmonic field, and applied it to handle-based mesh deformation and Poisson-based mesh merging. The scalar field can control the influence region by varying a parameter, and save the effort for users to prescribe non-intuitive boundary constraints. However due to the time-consuming quadratic-program, the algorithm is not efficient for the interactive application. In the future, we will explore more efficient (approximate) solutions, *e.g.* utilizing ADMM method (Yang and Zhang, 2011) for acceleration and seek more applications of the localized quasi-(bi)harmonic field.

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